Parameterized Complexity of the Anchored k-Core Problem for Directed Graphs*

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Abstract

Motivated by the study of unraveling processes in social networks, Bhawalkar, Kleinberg, Lewi, Roughgarden, and Sharma [ICALP 2012] introduced the ANCHORED k-CORE problem, where the task is for a given graph G and integers b, k, and p to find an induced subgraph H with at least p vertices (the core) such that all but at most b vertices (called anchors) of H are of degree at least k. In this paper, we extend the notion of k-core to directed graphs and provide a number of new algorithmic and complexity results for the directed version of the problem. We show that

- The decision version of the problem is NP-complete for every $k \geq 1$ even if the input graph is restricted to be a planar directed acyclic graph of maximum degree at most $k+2$.
- The problem is fixed parameter tractable (FPT) parameterized by the size of the core p for $k=1$, and W[1]-hard for $k \geq 2$.
- When the maximum degree of the graph is at most Δ, the problem is FPT parameterized by $p+\Delta$ if $k \geq \frac{\Delta}{2}$.

1 Introduction

The anchored k-core problem can be explained by the following illustrative example. We want to organize a workshop on Theory of Social Networks. We send invitations to most distinguished researchers in the area and received many replies of the following nature: "Yes, in theory, I would be happy to come but my final decision depends on how many people I know will be

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there." Thus we have a list of tentative participants, but some of them can cancel their participation and we are afraid that the cancellation process may escalate. On the other hand, we also have limited funds to reimburse travel expenses for a small number of participants, which we believe, will guarantee their participation. Thus we want to "anchor" a small subset of participants whose guaranteed participation would prevent the unraveling process, and by fixing a small group we hope to minimize the number of cancellations, or equivalently, maximize the number of participants, or the core.

Unraveling processes are common for social networks where the behavior of an individual is often influenced by the actions of her/his friends. New events occur quite often in social networks: some examples are usage of a particular cell phone brand, adoption of a new drug within the medical profession, or the rise of a political movement in an unstable society. To estimate whether these events or ideas spread extensively or die out soon, one has to model and study the dynamics of *influence propagation* in social networks. Social networks are generally represented by making use of undirected or directed graphs, where the edge set represents the relationship between individuals in the network. Undirected graph model works fine for some networks, say Facebook, but the nature of interaction on some social networks such as Twitter is asymmetrical: the fact that user A follows user B does not imply that that user B also follows A.[1] In this case, it is more appropriate to model interactions in the network by **directed** graphs. We add a directed edge (u, v) if v follows u.

In this work we are interested in the model of *user engagement*, where each individual with less than k people to follow (or equivalently whose in-degree is less than k) drops out of the network. This process can be contagious, and may affect even those individuals who initially were linked to more than k people, say follow on Twitter. An extreme example of this was given by Schelling (see page 17 of [16]): consider a directed path on n vertices and let $k = 1$. The left-endpoint has in-degree zero, it drops out and now the in-degree of its only out-neighbor in the path becomes zero and it drops out as well. It is not hard to see that this way the whole network eventually drops out as the result of a *cascade of iterated withdrawals*. In general at the end of all the iterated withdrawals the remaining engaged individuals form a unique maximal induced subgraph whose minimum in-degree is at least k. This is called as the *k-core* and is a well-known concept in the theory of social networks. It was introduced by Seidman [18] and also been studied in various social sciences literature [7, 8].

Preventing Unraveling: The unraveling process described above in Schelling's example of a directed path can be highly undesirable in many

[1] The first author follows LeBron James on Twitter (and so do 8,017,911 other people), but he only follows 302 people with the first author not being one of them.

scenarios. How can one attempt to prevent this unraveling? In Schelling's example it is easy to see: if we "buy" the left end-point person into being engaged then the whole path becomes engaged. In general we overcome the issue of unraveling by allowing some "anchors": these are the vertices that remain engaged irrespective of their payoffs. This can be achieved by giving them extra incentives or discounts. The hope is that with a few *anchors* we can now ensure a large subgraph remains engaged. This subgraph is called as the *anchored k-core*: each non-anchor vertex in this induced subgraph must have in-degree at least k while the anchored vertices can have arbitrary in-degrees. The problem of identifying k-cores in a network also has the following game-theoretical interpretation introduced by Bhawalkar et al. [2]: each user in the social network pays a cost of k to remain engaged. On the other hand, he/she receives a profit of one from every neighbor who is engaged. The "network effects" come into play, and an individual decides to remain engaged if has non-negative payoff, i.e., it has at least k in-neighbors who are engaged. The k-core can be viewed as the unique maximal equilibrium in this model.

Bhawalkar et al. [2] introduced the Anchored k-Core problem for (undirected) graphs. In the Anchored k-Core problem the input is an undirected graph $G = (V, E)$ and integers b, k, and the task is to find an induced subgraph H of maximum size with all vertices but at most b (which are anchored) to be of degree at least k. In this work we extend the notion of anchored k-core to directed graphs. We are interested in the case, when in-degrees of all but b vertices of H are at least k. More formally, we study the following parameterized version of the problem.

Directed Anchored k-Core (Dir-AKC)
Input: A directed graph $G = (V, E)$ and integers b, k, p.
Parameter 1: b.
Parameter 2: k.
Parameter 3: p.
Question: Do there exist sets of vertices $A \subseteq H \subseteq V(G)$ such that $|A| \leq b$, $|H| \geq p$, and every $v \in H \setminus a$ satisfies $d^-_{G[H]}(v) \geq k$?

We will call the set A as the *anchors*, the graph H as the *anchored k-core*. Note that the undirected version of Anchored k-Core problem can be modeled by the directed version: simply replace each edge $\{u, v\}$ by arcs (u, v) and (v, u). Keeping the parameters b, k, p unchanged it is now easy to see that the two instances are equivalent.

Parameterized Complexity: We are mainly interested in the parameterized complexity of Anchored k-Core. For the general background, we refer to the books by Downey and Fellows [10], Flum and Grohe [13] and Niedermeier [15]. Parameterized complexity is basically a two dimensional framework for studying the computational complexity of a problem. One

dimension is the input size n and another one is a parameter k. A problem is said to be *fixed parameter tractable* (or FPT) if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f. A problem is said to be in XP, if it can be solved in time $O(n^{f(k)})$ for some function f. The W-hierarchy is a collection of computational complexity classes: we omit the technical definitions here. The following relation is known amongst the classes in the W-hierarchy: $\text{FPT} = \text{W}[0] \subseteq W[1] \subseteq W[2] \subseteq \ldots$. It is widely believed that $FPT \neq W[1]$, and hence if a problem is hard for the class $W[i]$ (for any $i \geq 1$) then it is considered to be fixed-parameter intractable.

Previous Results: Bhawalkar et al. [2] initiated the algorithmic study of Anchored k-Core on undirected graphs and obtained an interesting dichotomy result: the decision version of the problem is solvable in polynomial time for $k \leq 2$ and is NP-complete for all $k \geq 3$. For $k \geq 3$, they also studied the problem from the viewpoint of parameterized complexity and approximation algorithms. The current set of authors [5] improved and generalized these results by showing that for $k \geq 3$ the problem remains NP-complete even on planar graphs.

Our Results: In this paper we provide a number of new results on the algorithmic complexity of Directed Anchored k-Core (Dir-AKC). We start (Section 2) by showing that that the decision version of Dir-AKC is NP-complete for every $k \geq 1$ even if the input graph is restricted to be a planar directed acyclic graph (DAG) of maximum degree at most $k+2$. Note that this shows that the directed version is in some sense strictly harder than the undirected version since it is known be in P if $k \leq 2$, and NP-complete if $k \geq 3$ [2]. The NP-hardness result for Dir-AKC motivates us to make a more refined analysis of the Dir-AKC problem via the paradigm of parameterized complexity. In Section 3, we obtain the following dichotomy result: Dir-AKC is FPT parameterized by p if $k = 1$, and W[1]-hard if $k \geq 2$. This fixed-parameter intractability result parameterized by p forces us to consider the complexity on special classes of graphs such as bounded-degree directed graphs or directed acyclic graphs. In Section 4, for graphs of degree upper bounded by Δ, we show that the Dir-AKC problem is FPT parameterized by $p + \Delta$ if $k \geq \frac{\Delta}{2}$. In particular, it implies that Dir-AKC is FPT parameterized by p for directed graphs of maximum degree at most four. We complement these results by showing in Section 5 that if $k < \frac{\Delta}{2}$ and $\Delta \geq 3$, then Dir-AKC is W[2]-hard when parameterized by the number of anchors b even for DAGs, but the problem is FPT when parameterized by $\Delta + p$ for DAGs of maximum degree at most Δ. Note that we can always assume that $b \leq p$, and hence any FPT result with parameter b implies FPT result with parameter p as well. On the other side, any hardness result with respect to p implies the same hardness with respect to b.

2 Preliminaries

We consider finite directed and undirected graphs without loops or multiple arcs. The vertex set of a (directed) graph G is denoted by $V(G)$ and its edge set (arc set for a directed graph) by $E(G)$. The subgraph of G induced by a subset $U \subseteq V(G)$ is denoted by $G[U]$. For $U \subset V(G)$ by $G - U$ we denote the graph $G[V(G) \setminus U]$. For a directed graph G, we denote by G^* the undirected graph with the same set of vertices such that $\{u, v\} \in E(G^*)$ if and only if $(u, v) \in E(G)$. We say that G^* is the *underlying* graph of G.

Let G be a directed graph. For a vertex $v \in V(G)$, we say that u is an *in-neighbor* of v if $(u, v) \in E(G)$. The set of all in-neighbors of v is denoted by $N_G^-(v)$. The *in-degree* $d_G^-(v) = |N_G^-(v)|$. Respectively, u is an *out-neighbor* of v if $(v, u) \in E(G)$, the set of all out-neighbors of v is denoted by $N_G^+(v)$, and the *out-degree* $d_G^+(v) = |N_G^+(v)|$. The *degree* $d_G(v)$ of a vertex v is the sum $d_G^-(v) + d_G^+$, and the *maximum degree* of G is $\Delta(G) = \max_{v \in V(G)} d_G(v)$. A vertex v of $d_G^-(v) = 0$ is called a *source*, and if $d_G^+(v) = 0$, then v is a *sink*. Observe that isolated vertices are sources and sinks simultaneously.

Let G be a directed graph. For $u, v \in V(G)$, it is said that v can be *reached* (or *reachable*) from u if there is a directed $u \to v$ path in G. Respectively, a vertex v can be reached from a set $U \subseteq V(G)$ if v can be reached from some vertex $u \in U$. Notice that each vertex is reachable from itself. We denote by $R_G^+(u)$ ($R_G^+(U)$ respectively) the set of vertices that can be reached from a vertex u (a set $U \subseteq V(G)$ respectively). Let $R_G^-(u)$ denoted the set of all vertices v such that u can be reached from v.

For two non-adjacent vertices s, t of a directed graph G, a set $S \subseteq V(G) \setminus \{s, t\}$ is said to be a *$s - t$ separator* if $t \notin R_{G-S}^+(s)$. An $s - t$ separator S is *minimal* if no proper subset $S' \subset S$ is a $s - t$ separator.

The notion of important separators was introduced by Marx [14] and generalized for directed graphs in [6]. We need a special variant of this notion. Let G be a directed graph, and let s, t be non-adjacent vertices of G. An minimal $s - t$ separator is an *important $s - t$ separator* if there is no $s - t$ separator S' with $|S'| \leq |S|$ and $R_{G-S}^-(t) \subset R_{G-S'}^-(t)$. The following lemma is a variant of Lemma 4.1 of [6]. Notice that to obtain it, we should replace the directed graph in Lemma 4.1 of [6] by the graph obtained from it by reversing direction of all arcs.

Lemma 1 ([6]). *Let G be a directed graph with n vertices, and let s, t be non-adjacent vertices of G. Then for every $h \geq 0$, there are at most 4^h important $s - t$ separators of size at most h. Furthermore, all these separators can be enumerated in time $O(4^h \cdot n^{O(1)})$.*

As further we are interested in the parameterized complexity of Dir-AKC, we show first NP-hardness of the problem.

Theorem 1. *For any $k \geq 1$, Dir-AKC is NP-complete, even for planar DAGs of maximum degree at most $k + 2$.*

Proof. We reduce SATISFIABILITY:

Satisfiability
Input: Sets of Boolean variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
Question: Can the formula $\phi = C_1 \vee \ldots \vee C_m$ be satisfied?

It is known (see e.g. [9]) that this problem remains NP-hard even if each clause contains at most 3 literals (notice that clauses of size one or two are allowed), each variable is used in at most 3 clauses: at least once in positive and at least once in negation, and the graph that correspond to a boolean formula is planar. Consider an instance of SATISFIABILITY with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$ that satisfies these restrictions on planarity and the number of occurrences of the variables. We construct the graph G as follows.

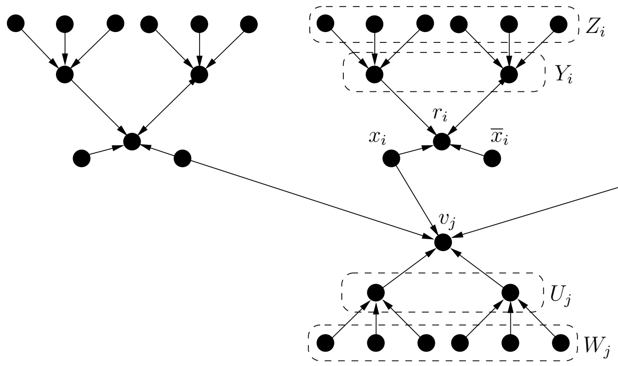

Figure 1: Construction of G for $k = 3$.

- For each $i \in \{1, \ldots, n\}$,
  - add vertices $x_i, \overline{x}_i, r_i$ and add arcs $(x_i, r_i), (\overline{x}_i, r_i)$;
  - add a set of $k - 1$ vertices Y_i and draw an arc from each of them to r_i;
  - for each vertex $y \in Y_i$, add k vertices and draw an arc from each of them to y, denote the set of these $k(k-1)$ vertices Z_i.
- For each $j \in \{1, \ldots, j\}$,
  - add a vertex v_j, and for each literal x_i ($\overline{x}_i$ respectively) in the clause C_j, join the vertex x_i ($\overline{x}_i$ respectively) with v_j by an arc;
  - add a set of $k - 1$ vertices U_j and draw an arc from each of them to v_j;
  - for each vertex $u \in U_j$, add k vertices and draw an arc from each of them to u, denote the set of these $k(k-1)$ vertices W_j.

Notice that if $k = 0$, then $Y_i = Z_i = U_j = W_j = \emptyset$. The construction of G is shown in Fig. 1. We set $b = n(k(k-1)+1) + mk(k-1)$ and $p = n((k+1)(k-1)+2) + m((k+1)(k-1)+1)$. It is straightforward to see that G is acyclic. Because each variable x_i is used at most 2 times in positive and at most 2 times in negations, $d_G(x_i), d_G(\overline{x}_i) \leq 3$ for all $i \in \{1, \ldots, n\}$, and $\Delta(G) \leq k+2$. Since $d_G(x_i) = 1$ or $d_G(\overline{x}_i) \leq 1$ for each $i \in \{1, \ldots, n\}$, G is planar.

We claim that all clauses $C_1, \ldots, C_m$ can be satisfied if and only if there are a set $A \subseteq V(G)$ and an induced subgraph H of G such that $A \subseteq V(H)$, $|A| \leq b, |V(H)| \geq p$ and for every $v \in V(H) \setminus A$ we have $d^-_H(v) \geq k$.

Suppose that we have a YES-instance of SATISFIABILITY and consider a truth assignment of $x_1, \ldots, x_n$ such that all clauses are satisfied. We construct A by including all the vertices $Z_1 \cup \ldots \cup Z_n \cup W_1 \cup \ldots \cup W_m$ in this set, and for each $i \in \{1, \ldots, n\}$, if $x_i =$ true, then x_i is included in A and $\overline{x}_i$ is included otherwise. Clearly, $|A| = |Z_1| + \ldots + |Z_n| + |W_1| + \ldots + |W_m| + n = n(k(k-1)+1) + mk(k-1) = b$. Let $H = G[A \cup Y_1 \cup \ldots \cup Y_n \cup U_1 \cup \ldots U_m \cup \{r_1, \ldots, r_n\} \cup \{v_1, \ldots, v_m\}]$. Consider $w \in V(H) \setminus A$. If $w \in Y_i$ for $i \in \{1, \ldots, n\}$, then w has k in-neighbors in $Z_i \subseteq A$. If $w = r_i$ for $i \in \{1, \ldots, n\}$, then w has $k-1$ in-neighbors in Y_i and either x_i or $\overline{x}_i$ is an in-neighbor of w as well. If $w \in U_j$ for $j \in \{1, \ldots, m\}$, then w has k in-neighbors in $W_j \subseteq A$. Finally, if $w = v_j$ for some $j \in \{1, \ldots, m\}$, then w has $k-1$ in-neighbors in U_j. As the clause C_j is satisfied, it contains a literal x_i or $\overline{x_i}$ that has the value true. Then by the construction of A, the corresponding vertex x_i or $\overline{x}_i$ respectively is in A, and w has one in-neighbor in A. It remains to observe that $|V(H)| = |A| + |Y_1| + \ldots + |Y_n| + |U_1| + \ldots + |U_m| = n(k(k-1)+1) + mk(k-1) + k(n+m) = p$.

Assume now there are a set $A \subseteq V(G)$ and an induced subgraph H of G such that $A \subseteq V(H)$, $|A| \leq b, |V(H)| \geq p$ and for every $v \in V(H) \setminus A$ we have $d^-_H(v) \geq k$.

Let $S = \{w \in V(G) \mid d^-_G(w) = 0\} = (\cup_{i=1}^n \{x_i, \overline{x_i}\}) \cup (\cup_{i=1}^n Z_i) \cup (\cup_{j=1}^m W_j)$ and $T = \{w \in V(G) \mid d^+_G(w) = 0\} = V(G) \setminus S = \{r_1, \ldots, r_n\} \cup (\cup_{i=1}^n Y_i) \cup (\cup_{j=1}^m U_j)$. We claim that $A \subseteq S$ and $T \subseteq H$. To show it, observe that any vertex $w \in S$ is in H if and only if $w \in A$ as $d^-_G(w) = 0$. Because $|V(G)| - |V(H)| \leq n$, at least $|S| - n$ vertices of S are in A. Since $|S| = b+n$, we conclude that exactly $b = |S| - n$ vertices of S are in A and $A \subseteq S$. Moreover, $V(H) = T \cup A$.

Let $z \in Z_i$ for some $i \in \{1, \ldots, n\}$ and assume that z is adjacent to $y \in Y_i$. If $z \notin A$, then $y \in T$ has at most $k-1$ in-neighbors in H, a contradiction. Hence, $Z_1 \cup \ldots \cup Z_n \subseteq A$. By the same arguments we conclude that $W_1 \cup \ldots \cup W_m \subseteq A$. Then we have exactly n elements of A in $\cup_{i=1}^n \{x_i, \overline{x}_i\}$. Consider a pair of vertices $x_i, \overline{x}_i$ for $i \in \{1, \ldots, n\}$. If $x_i, \overline{x}_i \notin A$, then $r_i \in T$ has at most $k-1$ in-neighbors in H, a contradiction. Therefore, for each $i \in \{1, \ldots, n\}$, exactly one vertex from the pair $x_i, \overline{x}_i$ is

in A. For $i \in \{1, \ldots, n\}$, we set the variable $x_i = \text{true}$ if the vertex $x_i \in A$, and $x_i = \text{false}$ otherwise.

It remains to prove that we have a satisfying truth assignment. Consider a clause C_j for $j \in \{1, \ldots, m\}$. The vertex $v_j \in T$ has $k-1$ in-neighbors in H that are vertices of T. Hence, it has at least one in-neighbor in A. It can be either a vertex x_i or $\overline{x}_i$ that correspond to a literal in C_j. It is sufficient to observe that if $x_i \in A$, then the literal $x_i = \text{true}$, and if $\overline{x}_i \in A$, then the literal $\overline{x}_i = \text{true}$ by our assignment. □

We conclude this section by the simple observation that Dir-AKC is in XP when parameterized by the number of anchors b. For a directed graph G with n vertices, we can consider all the at most n^b possibilities to choose the anchors, and then recursively delete non-anchor vertices that have the in-degree at most $k-1$. Trivially, if we obtain a directed graph with at least p vertices for some selection of the anchors, we have a solution and otherwise we can answer NO.

3 Dir-AKC parameterized by the size of the core

In this section we consider the Dir-AKC problem for fixed k when p is a parameter and obtain the following dichotomy: If $k = 1$ then the Dir-AKC problem is FPT parameterized by p, otherwise for $k \geq 2$ it is W[1]-hard parameterized by p.

Theorem 2. *For $k = 1$, the* Dir-AKC *problem is solvable in time $2^{O(p)} \cdot n^2 \log n$ on digraphs with n vertices.*

Proof. The proof is constructive, and we describe an FPT algorithm for the problem. Without loss of generality, we assume that $b < p \leq n$.

We apply the following preprocessing rule reducing the instance to an acyclic graph. Let $C_1, \ldots, C_r$ be strongly connected components of G. By making use of Tarjan's algorithm [19], the sets $C_1, \ldots, C_r$ can be found in linear time. Let $R = R^+_G\left(\bigcup_{i=1}^r V(C_i)\right)$ be the set of vertices reachable from strongly connected components. Then every $v \in R$ satisfies $d^-_{G[R]}(v) \geq 1$. If $b \geq p - |R|$, then we select in $V(G) \setminus R$ any arbitrary $b' = p - |R|$ vertices $a_1, \ldots, a_{b'}$. In this case we output the set of anchors $A = \{a_1, \ldots, a_{b'}\}$ and graph $H = G[A \cup R]$. Otherwise, if $b < p - |R|$, we set $G' = G - R$ and $p' = p - |R|$ and consider a new instance of Dir-AKC with the graph G' and the parameter p'.

To see that the rule is safe, it is sufficient to observe that a set of anchors A and a subgraph H' of size at least p' is a solution of the obtained instance if and only if $(A, H = G[V(H') \cup R])$ is a solution for the original problem. Let us remark that the preprocessing rule can be easily performed in time $O(n^2)$.

From now we can assume that G has no strongly connected components, i.e., G is a directed acyclic graph. Denote by $S = \{s_1, \ldots, s_h\}$ the set of sources of G. If $|S| \leq b$, then set $A = S$. In this case, we output the pair $(A, H = G)$. The pair (A, H) is a solution because every vertex $v \in V(G) \setminus S$ satisfies $d^-_G(v) \geq 1$. It remains to consider the case when $|S| > b$. For $i \in \{1, \ldots, h\}$, let $R_i = R^+_G(s_i)$. Then $V(G) = R^+_G(S) = \bigcup_{i=1}^h R_i$. Without loss of generality, we can assume that every anchored vertex is from S. Indeed, if s_i is an anchor, then each vertex of R_i can be included in a solution. Hence for every anchor $a \in R_j \setminus \{s_j\}$, we can delete anchor from a and anchor s_j, if it is not yet anchored. Since we can choose anchors only from S, we are able to reduce the problem to Partial Set Cover.

Partial Set Cover *Input* : A collection $X = \{X_1, \ldots, X_r\}$ of subsets of a finite n-element set U and positive integers p, b. *Parameter*: p. *Question*: Are there at most b subsets $X_{i_1}, \ldots, X_{i_b}$, $1 \leq i_1 < \ldots < i_b \leq r$, covering at least p elements of U, i.e., $\lvert \bigcup_{j=1}^b X_{i_j} \rvert \geq p$?

Bläser [3] showed that Partial Set Cover if FPTparameterized by p and can be solved in time $O(2^{O(p)} \cdot rn \log n)$. For Dir-AKC, we consider the collection of subsets $\{R_1, \ldots, R_r\}$ of $V(G)$. If we can select at most b subsets $R_{i_1}, \ldots, R_{i_b}$ such that $\lvert \cup_{j=1}^b R_{i_j} \rvert \geq p$, we return the solution with anchors $A = \{s_{i_1}, \ldots, s_{i_b}\}$ and $H = G[\bigcup_{j=1}^b R_{i_j}]$. Otherwise, we return a NO-answer.

Because our preprocessing can be done in time $O(n^2)$ and Partial Set Cover is solvable in time $2^{O(p)} \cdot n^2 \log n$, we conclude that the total running time is $2^{O(p)} \cdot n^2 \log n$. □

Now we complement Theorem 2 by showing that for $k \geq 2$, Dir-AKC becomes hard parameterized by the core size.

Theorem 3. *For any fixed $k \geq 2$, the* Dir-AKC *problem is* W[1]*-hard parameterized by p, even for DAGs.*

Proof. We reduce from the b-Clique problem which is known to be W[1]-hard [11]:

b-Clique *Input*: A undirected graph G and a positive integer b. *Parameter*: b *Question*: Is there a clique of size b in G?

From a given graph G we construct a directed graph G' as follows.

- Construct a copy of $V(G)$.

- For each edge $\{u, v\} \in E(G)$, construct a new vertex w_{uv} and join w_{uv} with u, v in the copy of $V(G)$ by arcs.

- Construct $k-2$ vertices $z_1, \ldots, z_{k-2}$, and for each $e \in E(G)$, join $z_1, \ldots, z_{k-2}$ with w_e by arcs.

It is straightforward to see that G' is a directed acyclic graph. We say that the vertex w_{uv} for $\{u, v\} \in E(G)$ is a *subdivision* vertex, and we say that $v \in V(G)$ is a *branch* vertex. Let $b' = b + k - 2$ and $p = \frac{b(b+1)}{2} + k - 2$. Let $Z = \{z_1 \ldots, z_{k-1}\}$. We claim that G has a clique of size b if and only if there is a set of at most b' vertices $A \subseteq V(G')$ such that there exists an an induced subgraph H of G' with at least p vertices, $A \subseteq V(H)$ and for any $v \in V(H) \setminus A$ we have $d_H^-(v) \geq k$.

Suppose that K is a clique in G of size b. We let $A = K \cup Z$ and define $U = \{w_{uv} | u, v \in K\}$. Notice that $|U| = \frac{b(b-1)}{2}$ and each vertex of U has two in-neighbors in $A \cap K$ and $k-2$ in-neighbors in Z. We conclude that $H = G'[A \cup U]$ has p vertices and for any $v \in V(H) \setminus A$ satisfies $d_H^-(v) \geq k$.

Assume now that there is a set of at most b' vertices $A \subseteq V[G']$ such that there exists an induced subgraph H of G' with at least p vertices, $A \subseteq V(H)$ and for any $v \in V(H) \setminus A$ we have $d_H^-(v) \geq k$. Since subdivision vertices of G' are sinks, we can assume that A contains only *branch* vertices and vertices from Z, as otherwise we can replace an anchor $a \in A$ that is a subdivision vertex of G' by an arbitrary branch vertex or a vertex of Z. Because branch vertices of G' and the vertices of Z are sources, any such vertex v is in H if and only if $v \in A$. Hence, H has at most b' sources of G' and at least $\frac{b(b-1)}{2}$ subdivision vertices. If there is a vertex $z_i \in Z$ such that $z_i \notin A$, then each subdivision vertex w_e has at most $k-1$ in-neighbors and H cannot contain subdivision vertices. Therefore $Z \subseteq A$ and A has at most $b' - (k-2) = b$ branch vertices. It remains to observe that a subdivision vertex w_{uv} has k in-neighbors in H if and only if $u, v \in A$. Then the claim follows. $\square$

4 Dir-AKC on graphs of bounded degree

In this section we show that Dir-AKC problem is FPT parameterized by $\Delta + p$ if $k \geq \frac{\Delta}{2}$.

In our algorithms we need to check the existence of solutions for Dir-AKC that have bounded size. It can be observed that if we are interested in solutions (A, H) such that $p \leq |V(H)| \leq q$, then for every positive q, we can express this problem in the first order logic. It was proved by Seese [17] that any graph problem expressible in the first-order logic can be solved in linear time on (directed) graphs of bounded degree. Later this result was extended for much more rich graph classes (see [12]). These meta theorems are very general, but do not provide good upper bounds for running time for particular problems. Hence, we give the following lemma. Our algorithms

use the random separation technique due to Cai et al. [4] (which is a variant of the color coding method introduced by Alon et al. [1]) .

Lemma 2. *There is a randomized algorithm with running time* $2^{O(\Delta q)} \cdot n$ *that for an instance of* Dir-AKC *with an* n*-vertex directed graph of maximum degree at most* Δ*, either returns a solution* (A, H) *with* $V(H) \geq p$ *or gives the answer that there is no solution with* $|V(H)| \leq q$*. Furthermore, the algorithm can be derandomized, and the deterministic variant runs in time* $2^{O(\Delta q)} \cdot n \log n$*.*

Proof. Consider an instance of Dir-AKC with an n-vertex directed graph G of maximum degree at most Δ. We assume that $b \leq p \leq n$. For given $q \geq p$, to decide if G contains a solution of size at most q, we do the following.

We color each vertex of G uniformly at random with probability $\frac{1}{2}$ by one of two colors, say red or blue. Let R be the set of vertices colored red. Observe that if there is a solution (A, H) with $|V(H)| \leq q$, then with probability at least $\frac{1}{2^q}$ all vertices of H are colored red and with probability at least $\frac{1}{2^{\Delta q}}$ all in- and out-neighbors of the vertices of H that are outside of H are colored blue. Using this observation, we assume that H is the union of some weakly connected components of the graph $G[R]$ induced by red vertices.

In time $O(\Delta n)$ we find all weakly connected components of $G[R]$. If there is a component C with at least $b + 1$ vertices of in-degree at most $k - 1$ (in C), then we discard this component as it cannot be a part of any solution. Denote by $C_1, \ldots, C_r$ the remaining components. For $i \in \{1, \ldots, r\}$, let $A_i = \{v \in V(C_i) | d^-_{C_i}(v) < k\}$, $b_i = |A_i|$ and $p_i = |V(C_i)|$.

Thus everything boils down to the problem of finding a set $I \subseteq \{1, \ldots, r\}$ such that $\sum_{i \in I} b_i \leq b$ and $\sum_{i \in I} p_i \geq p$. But this is the well known Knapsack problem, which is solvable in time $O(bn)$ by dynamic programming. If we obtain a solution I, then we output (A, H), where $A = \cup_{i \in I} A_i$ and $H = G[\cup_{i \in I} V(C_i)]$. Otherwise, we return a NO-answer. Notice that this algorithm can also find a solution (A, H) with $|V(H)| > q \geq p$.

It remains to observe that for any positive number $\alpha < 1$, there is a constant c_α such that after running our randomized algorithm $c_\alpha \cdot 2^{\Delta q}$ times, we either find a solution (A, H) or can claim that with probability α that it does not exist.

This algorithm can be derandomized by the technique proposed by Alon et al. [1]: replace the random colorings by a family of at most $2^{O(\Delta q)} \cdot \log n$ hash functions which are known to be constructible in time $2^{O(\Delta q)} \cdot n \log n$. □

Our next aim is to prove that for $k > \Delta/2$ the Dir-AKC problem is FPT when parameterized by the number of anchors b.

Lemma 3. *Let Δ be a positive integer. If $k > \Delta/2$, then the* Dir-AKC *problem can be solved in time $2^{O(\Delta^2 b)} \cdot n \log n$ for n-vertex directed graphs of maximum degree at most Δ.*

Proof. Suppose (A, H) is a solution for the Dir-AKC problem. Let us observe that because $k > \Delta/2$, for every vertex $v \in V(H) \setminus A$, we have $d^-_H(v) > d^+_H(v)$. Recall that for any directed graph, the sum of in-degrees equals the sum of out-degrees. Then

$$\sum_{v \in V(H) \setminus A} (d^-_H(v) - d^+_H(v)) = \sum_{v \in A} (d^+_H(v) - d^-_H(v)).$$

Since for every vertex $v \in V(H) \setminus A$, $d^-_H(v) - d^+_H(v) \geq 1$, we have that

$$|V(H) \setminus A| \leq \sum_{v \in V(H) \setminus A} (d^-_H(v) - d^+_H(v)).$$

On the other hand, $d^+_H(v) - d^-_H(v) \leq \Delta$, and we arrive at

$$|V(H) \setminus A| \leq \sum_{v \in V(H) \setminus A} (d^-_H(v) - d^+_H(v)) = \sum_{v \in A} (d^+_H(v) - d^-_H(v)) \leq \Delta |A|.$$

Hence, $|V(H)| \leq (\Delta + 1)|A| \leq (\Delta + 1)b$. Using this observation, we can solve the Dir-AKC problem as follows. If $p > (\Delta + 1)b$, then we return a NO-answer. If $p \leq (\Delta + 1)b$, we apply Lemma 2 for $q = (\Delta + 1)b$, and solve that problem in time $2^{O(\Delta^2 b)} \cdot n \log n$. $\square$

Now we show that if $k = \frac{\Delta}{2}$ then the Dir-AKC problem is FPT parameterized by $\Delta + p$. Due the space restrictions we only sketch the proof of the following lemma.

Lemma 4. *Let Δ be a positive integer. If $k = \Delta/2$, then the* Dir-AKC *problem can be solved in time $2^{O(\Delta^3 b + \Delta^2 bp)} \cdot n^{O(1)}$ for n-vertex directed graphs of maximum degree at most Δ.*

Proof. We describe an FPT algorithm. Consider an instance of the Dir-AKC problem. Without loss of generality we assume that $b < p \leq n$.

We apply the following preprocessing rule. Suppose that G has a (weakly) connected component C such that for any $v \in V(C)$, $d^-_C(v) = d^+_C(v) = k$. If $b \geq p - |V(C)|$, then we choose a set A of $b' = p - |V(C)|$ vertices arbitrary in $V(G) \setminus V(C)$. Then we return a YES-answer, as the anchors A and $H = G[A \cup V(C)]$ is a solution. Otherwise, if $b < p - |V(C)|$, we let $G' = G - V(C)$ and $p' = p - |V(C)|$. Now we consider a new instance of the problem with the graph G' and the parameter p'. To see that the rule is safe, it is sufficient to observe that a set of anchors A and a subgraph H' of size at least p' is a solution of the obtained instance if and only if A and

$H = G[V(H') \cup V(C)]$ is a solution for the original problem. From now we assume that G has no such components.

We need the following claim.

Claim A. *If an instance of the* Dir-AKC *problem has a core with at least $(\Delta p + 1)b + 1$ vertices, then it has a solution (A, H) with the following property: there is a vertex $t \in V(H) \setminus A$ reachable in H from any vertex of H. Moreover, for each vertex v of H, there is a path from v to t with all vertices except v in $V(H) \setminus A$.*

Proof of Claim A. Let (A, H') be a solution with the set of anchors A and such that $V(H') > (\Delta p + 1)b$.

We show that $V(H') = R^+_{H'}(A)$, i.e., all vertices of H' are reachable from the anchors. To obtain a contradiction, suppose that there is a vertex $u \in V(H')$ such that $u \notin R^+_{H'}(A)$. Let $U = R^-_{H'}(u)$, i.e., U is the set of vertices from which we can reach u. Clearly, $A \cap U = \emptyset$. Therefore, $d^-_{H'}(v) \geq k = \Delta/2$ for $v \in U$. Notice that for a vertex $v \in U$, $N_{H'}(v) \subseteq U$ by the definition. Hence, $d^-_{G[U]}(v) \geq k = \Delta/2$ for $v \in U$. Because the sum of in-degrees equals the sum of out-degrees, for every vertex $v \in U$, we have that $d^-_{G[U]}(v) = d^+_{G[U]}(v) = k = \Delta/2$. Then $C = G[U]$ is a component of G such that for every $v \in V(C)$, $d^-_C(v) = d^+_C(v) = k$, but such components are excluded by the preprocessing; a contradiction.

Observe now that if $d^-_{H'}(v) < d^+_{H'}(v)$, then $d^-_{H'}(v) < k$ and thus $v \in A$. Hence, by adding at most Δb (maybe multiple) arcs from $V(H') \setminus A$ to A, joining the vertices $v \in V(H')$ of degrees $d^-_{H'}(v) > d^+_{H'}(v)$ with vertices of degrees $d^-_{H'}(v) < d^+_{H'}(v)$, we can transform H' into a disjoint union of directed Eulerian graphs. Since $V(H') = R^+_{H'}(A)$, each of these directed Eulerian graphs contains at least one vertex of A. Thus the set of arcs of H' can be covered by at most Δb arc-disjoint directed walks, each walk starting from a vertex of A and never coming back to A. Because $d^-_{H'}(v) \geq k$ for $v \in V(H') \setminus A$, we have that $|E(G')| \geq k(|V(H')| - b) > \Delta kbp$. Then there is a walk W with at least $kp + 1$ arcs. Let $a \in A$ be the first vertex of W and let t be the last vertex of the walk. The walk W visits a only once, t and all other vertices of W are visited at most k times. We conclude that W has at least p vertices.

Let $R = R^-_{H'-A}(t)$ and let $A' = \{a \in A \mid N^+_{H'}(a) \cap R \neq \emptyset\}$. Consider $H = G[R \cup A']$. Since $V(W) \subseteq V(H)$, $|V(H)| \geq p$. For any $v \in V(H) \setminus A$, the in-neighbors of v in H' are in H by the construction and, therefore, $d^-_H(v) \geq k$. It remains to observe that to select at most b anchors, we take $A' \subseteq V(H)$. $\square$

Using Claim A, we proceed with our algorithm. We try to find a solution such that H has at most $q = (\Delta p + 1)b$ vertices by applying Lemma 2. It takes time $O(2^{O(\Delta^2 bp)} \cdot n \log n)$. If we obtain a solution, then we return it and

stop. Otherwise, we conclude that every core contains at least $(\Delta p+1)b+1$ vertices. By Claim A, we can search for a solution H with a non-anchor vertex t which is reachable from all other vertices of H by directed paths avoiding A. Notice that since t is a non-anchor vertex, we have that $d^-_G(t) \geq k$. We try at most n possibilities for all possible choices of t, and solve our problem for each choice. Clearly, if we get a YES-answer for one of the choices, we return it and stop. Otherwise, if we fail, we return a NO-answer.

From now we assume that we already selected t. We denote by G' the graph obtained from G by adding an artificial source vertex s joined by arcs with all the vertices $v \in V(G)$ with $d^-_G(v) < k$. Observe that $(s,t) \notin E(G')$.

Suppose that (A,H) is a solution with the set of anchors A such that $t \in V(H) \setminus A$ is reachable in H from any vertex of H by a path with all inner vertices in $V(H) \setminus A$. Denote by $\delta_{G'}(H)$ the set $\{v \in V(H) \mid N^-_{G'}(v) \setminus V(H) \neq \emptyset\}$, i.e., $\delta_{G'}(H)$ contains vertices that have in-neighbors outside H. We need a chain of claims about the structure of H in G'.

Claim B. *$|\delta_{G'}(H) \setminus A| \leq \Delta b$.*

Proof of Claim B. Let $X = \{v \in V(H) \mid d^-_H(v) \geq k$ and $d^+_H(v) < k\}$, $Y = \{v \in V(H) \mid d^-_H(v) = d^+_H(v) = k\}$ and $Z = \{v \in V(H) \mid d^-_H(v) < k\}$. Clearly,

$$\sum_{v\in X}(d^-_H(v) - d^+_H(v)) + \sum_{v\in Y}(d^-_H(v) - d^+_H(v)) = \sum_{v\in Z}(d^+_H(v) - d^-_H(v))$$

Observe that $d^-_H(v) - d^+_H(v) \geq 1$ for $v \in X$, $d^-_H(v) - d^+_H(v) = 0$ for $v \in Y$ and $d^+_H(v) - d^-_H(v) \leq \Delta$ for $v \in Z$. Hence, $|X| \leq \Delta|Z|$. If $d^-_H(v) < k$ for $v \in V(H)$, then $v \in A$. It follows that $Z \subseteq A$ and $|Z| \leq b$. We have $|X| \leq \Delta b$. Consider a vertex $v \in \delta_{G'}(H) \setminus A$. It has at least one in-neighbor outside H in G and $d^-_H(v) \geq k$. Then $d^+_H(v) < k$ and $v \in X$. We conclude that $\delta_{G'}(H) \setminus A \subseteq X$ and $|\delta_{G'}(H) \setminus A| \leq \Delta b$. $\square$

Claim C. *There is an $s-t$ separator S in G' of size at most $(\Delta(k-1)+1)b$ such that $V(H) \setminus A \subseteq R^-_{G'-S}(t)$.*

Proof of Claim C. Let $S = \Big(\delta_{G'}(H) \cap A\Big) \cup \Big(\bigcup_{v\in\delta_{G'}(H)\setminus A}(N^-_G(v) \setminus V(H)\Big)$, i.e., the set containing all anchors that are in $\delta_{G'}$, and for each non-anchor vertex of $\delta_{G'}$ containing all its in-neighbors outside of H. Consider a directed (s,t)-path P in G'. Let v be the first vertex in P that is in $V(H)$ and let u be its predecessor in P. If $v \in A$, then $v \in S$. If $v \notin A$, then $u \neq s$ as H has no non-anchor vertices with in-degree at most $k-1$ in G. Then $u \in S$. We conclude that each (s,t)-path contains a vertex of S, i.e., this set is an $s-t$ separator.

Observe that $V(H) \setminus A \subseteq R^-_{G'-S}(t)$ by the definition of S and the fact that t can be reached from any vertex of H in this graph by a path with all inner vertices in $V(H) \setminus A$.

It remains to show that $|S| \leq (\Delta(k-1)+1)b$. By Claim B, $|\delta_{G'}(H)\setminus A| \leq \Delta b$. A vertex $v \in \delta_{G'}(H) \setminus A$ has at least one out-neighbor in H because t is reachable from v. Then v has at most $k-1$ in-neighbors outside H. Hence $|S| \leq |A| + (k-1)(\delta_{G'}(H) \setminus A) \leq (\Delta(k-1)+1)b$. $\square$

Now we can prove the following claim about important $s-t$ separators in G'.

Claim D. *There is an important $s-t$ separator S^* of size at most $(\Delta(k-1)+1)b$ in G' such that $V(H) \subseteq R^-_{G'-S^*}(t) \cup S^*$.*

Proof of Claim D. By Claim C, there is an $s-t$ separator S' in G' of size at most $(\Delta(k-1)+1)b$ such that $V(H) \setminus A \subseteq R^-_{G'-S'}(t)$. Notice that S' not necessary a minimal separator, but there is a minimal $s-t$ separator $S \subseteq S'$. Clearly, $|S| \leq (\Delta(k-1)+1)b$.

We show that $V(H) \subseteq R^-_{G'-S}(t) \cup S$. Because $R^-_{G'-S'}(t) \subseteq R^-_{G'-S}(t)$, we have that $V(H) \setminus A \subseteq R^-_{G'-S}(t)$. Also if an anchor a is in $R^-_{G'-S'}(t)$, then $a \in R^-_{G'-S}(t)$. Let $a \in A \cap S'$. If $a \in A \cap S$, then $a \in R^-_{G'-S}(t) \cup S$. If $a \notin S$, then by Claim C, a has an out-neighbor $v \in R^-_{G'-S'}(t)$ and in this case we have $a \in R^-_{G'-S}(t)$.

It remains to observe that there is an important $s-t$ separator S^* such that $|S^*| \leq |S| \leq (\Delta(k-1)+1)b$ and $R^-_{G'-S}(t) \subseteq R^-_{G'-S^*}(t)$. Therefore, $V(H) \subseteq R^-_{G'-S}(t) \cup S \subseteq R^-_{G'-S^*}(t) \cup S^*$. $\square$

The next step of our algorithm is to check all important $s-t$ separators in G' of size at most $(\Delta(k-1)+1)$. By Lemma 1, there are at most $4^{(\Delta(k-1)+1)b}$ important $s-t$ separators and they can be listed in time $2^{O(\Delta^2 b)} \cdot n^c$. For each important $s-t$ separator S^*, we consider the set of vertices $U = R^-_{G'-S^*}(t) \cup S^*$ and decide whether there is a solution such that $V(H) \subseteq U$. If we have a solution for some S^*, then we return a YES-answer and stop. Otherwise, if we fail to find such a solution for all important separators, we use Claim D to deduce that there is no solution.

From now on, we assume that an important $s-t$ separator S^* is given and that $U = R^-_{G'-S^*}(t) \cup S^*$. In what follows, we describe a procedure of finding a solution with $V(H) \subseteq U$.

Denote by D the set $\{v \in U \mid d^-_G(v) > 0\}$. We need the following observation.

Claim E. *Set D contains at most $(\Delta+1)(\Delta(k-1)+1)b$ vertices.*

Proof of Claim E. Let $Q = G[U]$. Let $X = \{v \in V(Q) \mid d_Q^-(v) \geq k \text{ and } d_Q^+(v) < k\}$, $Y = \{v \in V(Q) \mid d_Q^-(v) = d_Q^+(v) = k\}$ and $Z = \{v \in V(Q) \mid d_Q^-(v) < k\}$. Clearly,

$$\sum_{v \in X}(d_Q^-(v) - d_Q^+(v)) + \sum_{v \in Y}(d_Q^-(v) - d_Q^+(v)) = \sum_{v \in Z}(d_Q^+(v) - d_Q^-(v))$$

Observe that $d_Q^-(v) - d_Q^+(v) \geq 1$ for $v \in X$, $d_Q^-(v) - d_Q^+(v) = 0$ for $v \in Y$ and $d_Q^+(v) - d_Q^-(v) \leq \Delta$ for $v \in Z$. Hence, $|X| \leq \Delta|Z|$.

Recall that G' is obtained from G by joining s with all vertices of in-degree at most $k-1$. Since S^* is an $s-t$ separator, if for $v \in U$, $d_Q^-(v) < k$, then $v \in S^*$. Hence, $Z \subseteq S^*$ and $|Z| \leq |S^*| \leq (\Delta(k-1)+1)b$. If for for $v \in U$, $d_G^-(v) > k$, then $v \in X \cup Z$. We conclude that $|D| \leq |X| + |Z| \leq (\Delta+1)|Z| \leq (\Delta+1)(\Delta(k-1)+1)b$. $\square$

Recall that set $\delta_{G'}(H)$ contains vertices of H that have in-neighbors outside of H. If $v \in \delta_{G'}(H) \setminus A$, then it has at least k in-neighbors in H and at least one in-neighbor outside H. Notice that $s \notin N_{G'}^-(v)$ because $d_G^-(v) \geq d_H^-(v) \geq k$. Hence, $d_G^-(v) > k$. Because $V(H) \subseteq U$, $\delta_{G'}(H) \setminus A \subseteq D$. By Claim C, $|\delta_{G'}(H) \setminus A| \leq \Delta b$, and by Claim E, $|D| \leq (\Delta+1)(\Delta(k-1)+1)b$. We consider all at most $2^{(\Delta+1)(\Delta(k-1)+1)b}$ possibilities to select $\delta_{G'}(H) \setminus A$. For each choice of $\delta_{G'}(H) \setminus A$, we guess the arcs that join the vertices that are outside H with the vertices of $\delta_{G'}(H) \setminus A$ and delete them. Denote the graph obtained from G by F. Recall that from each vertex v of $\delta_{G'}(H) \setminus A$, there is a directed path to t that avoids A. Hence, v has at least one out-neighbor in H and at most $\Delta - 1$ in-neighbors in G. Also v has at least k in-neighbors in H, and we delete at most $d_G^-(v) - k$ arcs. Therefore, for v we choose at most $k-1$ arcs out of at most $\Delta - 1$ arcs. We can upper bound the number of possibilities for v by $2^{\Delta-1}$, and the total number of possibilities for $\delta_{G'}(H) \setminus A$ is $2^{(\Delta-1)\Delta b}$.

Observe that (A, H) is a solution for the new instance of Dir-AKC, where G is replaced by F for a correct guess of the deleted arcs. Also each solution for the new instance provides a solution for the graph G, because if we put deleted arcs back, then we can only increase in-degrees. Hence, we can check for each possible choice of the set of deleted arcs, whether the new instance has a solution. If for some choice we obtain a solution, then we return a YES-answer. Otherwise, if we fail for all choices, then we return a NO-answer. Further we assume that F is given.

Denote by F' the graph obtained from F by the addition of a vertex s joined by arcs with all the vertices $N_{G'}^+(s)$. Now $\delta_{F'}(H) = \{v \in V(H) \mid N_{F'}^-(v) \setminus V(H) \neq \emptyset\}$. By the choice of F, $\delta_{F'}(H) = \delta_{G'}(H) \cap A$ and, therefore, $|\delta_{F'}(H)| \leq b$. Also $\delta_{F'}(H)$ is an $s-t$ separator in F' by Claim C.

Now we can prove the following.

Claim F. *There is an important $s-t$ separator $\hat{S}$ of size at most b in F' such that $(\hat{S}, G[R^-_{F'-\hat{S}}(t) \cup \hat{S}])$ is a solution for the instance of the* Dir-AKC *problem for the graph G.*

Proof of Claim F. Let $U = R^-_{F'-\hat{S}}(t) \cup \hat{S}$. It was already observed that $\delta^*_{G'}(H)$ is an $s-t$ separator in F' of size at most b. Then there is a minimal $s-t$ separator $S \subseteq \delta^*_{G'}(H)$. Clearly, $|S| \leq b$.

As before in the proof of Claim D, we show that $V(H) \subseteq R^-_{F'-S}(t) \cup S$. Because for any vertex v of H, there is a directed (v,t) path with all inner vertices in $V(H) \setminus A$, $V(H) \setminus A \subseteq R^-_{F'-\delta_{F'}(H)}(t)$. Because $R^-_{F'-\delta_{F'}(H)}(t) \subseteq R^-_{F'-S}(t)$ we have $V(H) \setminus A \subseteq R^-_{F'-S}(t)$. Also if $a \in A$ is in $R^-_{F'-\delta_{F'}(H)}(t)$, then $a \in R^-_{F'-S}(t)$. Let $a \in A \cap \delta_{F'}(H)$. Trivially, if $a \in A \cap S$, then $a \in R^-_{F'-S}(t) \cup S$. If $a \notin S$, then a has an out-neighbor $v \in R^-_{F'-\delta_{F'}(H)}(t)$ and $a \in R^-_{F'-S}(t)$. Then there is an important $s-t$ separator $\hat{S}$ such that $|\hat{S}| \leq |S| \leq b$ and $R^-_{F'-S}(t) \subseteq R^-_{F'-\hat{S}}(t)$. Therefore, $V(H) \subseteq R^-_{F'-S}(t) \cup S \subseteq R^-_{F'-S^*}(t) \cup S^*$, and $|U| \geq p$.

It remains to observe that s is adjacent to all vertices of G with in-degrees at most $k-1$ and S^* is an $s-t$ separator. It immediately follows that for any vertex $v \in R^-_{F'-S^*}(t)$, $d^-_{F(U)}(v) \geq k$. Then $(\hat{S}, G[R^-_{F'-\hat{S}}(t) \cup \hat{S}])$ is a solution. □

The final step of our algorithm is to enumerate all important $s-t$ separators $\hat{S}$ of size at most b in F', which number by Lemma 1 is at most 4^b, and for each $\hat{S}$, check whether $(\hat{S}, G[R^-_{F'-\hat{S}}(t) \cup \hat{S}])$ is a solution. Recall that all these separators can be listed in time $2^{O(b)} \cdot n^c$. We return a YES-answer if we obtain a solution for some important separator, and a NO-answer otherwise.

To complete the proof, let us observe that each step of the algorithm runs either in polynomial or FPT time. Particularly, the preprocessing is done in time $O(\Delta n)$. Then we check the existence of a solution of a bounded size in time $2^{O(\Delta^2 bp)} \cdot n \log n$. Further we consider at most n possibilities to choose t. For each t, we consider at most $4^{(\Delta(k-1)+1)b}$ important $s-t$ separators S^*. Recall, that they can be listed in time $2^{O(\Delta^2 b)} \cdot n^c$ for some constant c. Then for each S^*, we have at most $2^{(\Delta+1)(\Delta(k-1)+1)b+(\Delta-1)}$ possibilities to construct F, and it can be done in time $2^{O(\Delta^3 b)} + O(\Delta n)$. Finally, there are at most 4^b important $s-t$ separators $\hat{S}$ and they can be listed in time $2^{O(b)} \cdot n$ for some c. We conclude that the total running time is $2^{O(\Delta^3 b + \Delta^2 bp)} \cdot n^c$ for some constant c. □

Combining Lemmas 3 and 4, we obtain the following theorem.

Theorem 4. *Let Δ be a positive integer. If $k \geq \frac{\Delta}{2}$, then the* Dir-AKC *problem can be solved in time $2^{O(\Delta^3 b + \Delta^2 bp)} \cdot n^{O(1)}$ for n-vertex directed graphs of maximum degree at most Δ.*

Theorems 2 and 4 give the next corollary.

Corollary 1. *The* Dir-AKC *problem can be solved in time $2^{O(bp)} \cdot n^{O(1)}$ for n-vertex directed graphs of maximum degree at most* 4.

5 Conclusions

We proved that Dir-AKC is NP-complete even for planar DAGs of maximum degree at most $k+2$. It was also shown that Dir-AKC is FPT when parameterized by $p + \Delta$ for directed graphs of maximum degree at most Δ whenever $k \geq \Delta/2$. It is natural to ask whether the problem is FPT for other values k. This question is interesting even for the special case $\Delta = 5$ and $k = 2$.

For the special case of directed acyclic graphs (DAGs) we understand the complexity of the problem much better. Theorem 3 showed that Dir-AKC on DAGs is W[1]-hard parameterized by p for every fixed $k \geq 2$, when the degree of the graph is not bounded. We now show the following theorem that gives W[2]-hardness of Dir-AKC when parameterized by the number of anchors b (recall that we can always assume that $b \leq p$).

Theorem 5. *For any $\Delta \geq 3$ and any positive $k < \frac{\Delta}{2}$,* Dir-AKC *is* W[2]-*hard (even on DAGs) when parameterized by the number of anchors b on graphs of maximum degree at most Δ.*

Proof. First, we prove the claim for $k = 1$ and $\Delta = 3$. We reduce from the b-Set Cover problem which is known to be W[2]-hard [11]:

b-Set Cover
Input : A collection $X = \{X_1, \ldots, X_r\}$ of subsets of a finite n-element set U and a positive integer b.
Parameter: b
Question: Are there at most b subsets $X_{i_1}, \ldots, X_{i_b}$ such that these sets cover U, i.e., $U = \bigcup_{j=1}^{b} X_{i_j}$?

Let $U = \{u_1, \ldots, u_n\}$. We construct the directed graph G as follows (see Fig. 2).

- For $i \in \{1, \ldots, r\}$, assume that $X_i = \{u_{j_1}, \ldots, u_{j_s}\}$ and
  - construct a vertex v_i and s vertices $x_{ij_1}, \ldots, x_{ij_s}$;
  - construct arcs $(v_i, x_{ij_1}), (x_{ij_1}, x_{ij_2}), \ldots, (x_{ij_{s-1}}, x_{ij_s})$.

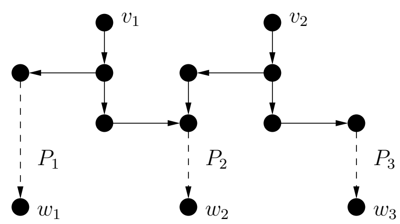

Figure 2: Construction of G for $U = \{u_1, u_2, u_3\}$ and $X_1 = \{u_1, u_2\}, X_2 = \{u_2, u_3\}$.

- For $j \in \{1, \ldots, n\}$, assume that u_j is included in the sets $X_{i_1}, \ldots, X_{i_t}$ and
  - construct a vertex w_j and t vertices $y_{ji_1}, \ldots, y_{ji_t}$;
  - construct arcs $(y_{ji_1}, y_{ji_2}), \ldots, (y_{ji_{t-1}}, y_{ji_t})$;
  - join y_{ji_t} with w_j by a directed path P_j of length $\ell = 2rn + r$.
- For $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n\}$, if $u_j \in X_i$, then construct an arc (x_{ij}, y_{ji}).

It is straightforward to see that G is a directed acyclic graph of maximum degree at most 3. We set $p = n\ell$. We claim that U can be covered by at most b sets if and only if there is a set of at most b vertices A such that there exists an induced subgraph H of G with at least p vertices, $A \subseteq V(H)$ and for any $v \in V(H) \setminus A$, $d^-_H(v) \geq 1$.

Notice that $v_1, \ldots, v_r$ are the sources of G, $w_1, \ldots, w_n$ are the sinks, and $V(G) = \bigcup_{i=1}^r R^+_G(v_i)$. Observe also that w_j can be reached from v_i if and only if $u_j \in X_i$.

Suppose that U can be covered by at most b sets say $X_{i_1}, \ldots, X_{i_b}$. Let $A = \{v_{i_1}, \ldots, v_{i_b}\}$ and $H = G[R^+_G(A)]$. It is straightforward to see that for any vertex $z \in V[H]$, $d^-_H(z) \geq 1$. Because U is covered, all vertices $w_1, \ldots, w_n$ are in H and, therefore, $V(P_1) \cup \ldots \cup V(P_n) \subseteq V(H)$. It remains to observe that $|V(P_1) \cup \ldots \cup V(P_n)| = n(\ell + 1) \geq p$ and we conclude that (A, H) is a solution of our instance of Dir-AKC.

Assume now that (A, H) is a solution of the Dir-AKC problem. Without loss of generality we can assume that that each $a \in A$ is a source of G. Otherwise, $a \in R^+_G(v_i)$ for some source v_i, and we can replace a by v_i in A (or delete it if $v_i \in A$ already). Let $\{i \mid 1 \leq i \leq n,\ v_i \in A\} = \{i_1, \ldots, i_b\}$. We show that $X_1, \ldots, X_{i_b}$ cover U. To obtain a contradiction, assume that there is an element $u_j \in U$ such that $u_j \notin X_{i_1} \cup \ldots \cup X_{i_b}$. Then the vertex w_j is not reachable from A. Hence, the vertices of P_j are not reachable from A. It follows that $V(P_j) \cap V(H) = \emptyset$. We have that $|V(H)| \leq |V(G)| - |V(P_j)|$. Because $|X_i| \leq n$ for $i \in \{1, \ldots, r\}$ and each u_h is included in at most r sets for $h \in \{1, \ldots, n\}$, $|V(G)| \leq r(n+1) + n(r + \ell) = 2rn + r + n\ell = 2rn + r + p$.

Therefore, $|V(H)| \leq p + (2rn + r - (\ell+1)) < p$ because P_j has $\ell+1$ vertices; a contradiction.

Now we prove W[2]-hardness for $k \geq 2$ and $\Delta > 2k$. We reduce from an instance of the Dir-AKC problem with $k = 1$ and $\Delta = 3$. Consider an instance of this problem with a directed acyclic graph G and positive integers b, p. Assume that $b \leq p \leq |V(G)|$ and $|V(G)| \geq 3$. We construct the graph G' as follows (see Fig. 3).

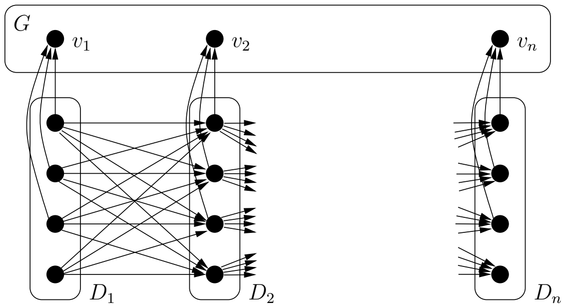

Figure 3: Construction of G' for $k = 4$.

- Construct a copy of G and denote its vertices by $v_1, \ldots, v_n$.
- For each $i \in \{1, \ldots, n\}$, construct a set of k vertices D_i and join $k-1$ vertices of this set with v_i by arcs.
- For each $i \in \{2, \ldots, n\}$, join each vertex of D_{i-1} with all vertices of D_i by arcs.

Clearly, G' is a directed acyclic graph. We let $b' = b + k$ and $p' = p + nk$. Let also $D = D_1 \cup \ldots \cup D_n$. Notice that for each $v \in V(G)$, $d_{G'}(v) = d_G(v) + k - 1 \leq k + 2 \leq \Delta$ as maximum degree of G is 3. For $v \in D$, $d_{G'}(v) \leq 2k + 1 \leq \Delta$. Hence maximum degree of G' is at most Δ. We now claim that there is a set of at most b vertices $A \subseteq V(G)$ such that there exists an an induced subgraph H of G with at least p vertices, $A \subseteq V(H)$ and for any $v \in V(H') \setminus A$, $d^-_H(v) \geq 1$ if and only if there is a set of at most b' vertices $A' \subseteq V(G')$ such that there exists an an induced subgraph H' of G' with at least p' vertices, $A' \subseteq V(H')$ and for any $v \in V(H) \setminus A$, $d^-_{H'}(v) \geq k$.

Suppose that our original instance of Dir-AKC has a solution (A, H). We let $A' = A \cup D_1$ and $H' = G'[V(H) \cup D]$. Then each vertex $v \in D \setminus A'$ has k in-neighbors in D. It remains to observe that each vertex v of G' from $V(G) \setminus A'$ has at least one in-neighbor in $V(G)$ and $k-1$ in-neighbors in D. Therefore, $d^-_{G'}(v) \geq k$.

Assume now that (A', H') is a solution for the constructed instance of Dir-AKC with $|A'| \leq b'$ and $|V(H)| \geq p'$. If $|D \cap A'| < k$, then we claim

that $D \cap V(H') \subseteq A'$. To prove it, suppose that $(V(H') \cap D) \setminus A \neq \emptyset$ and consider the smallest index i such that there is $v \in (V(H') \cap D_i) \setminus A$. Clearly, $i \geq 2$. The vertex v has in-neighbors only in D_{i-1}. By the choice of i, D_{i-1} has at most $k-1$ vertices of H', because they can be only anchors and $|D \cap A'| < k$. Then $d^-_{H'}(v) < k$, a contradiction.

Then if $|D \cap A'| < k$, $V(H') \subseteq V(G) \cup A'$ and $|V(H')| \leq n + b + k \leq n + p + k < p'$ as $n \geq 3$ and $k \geq 2$. This contradicts our assumption about size of H'. Hence, at least k anchors are in D and $|A' \setminus D| \leq b$. Let $A = A' \setminus D$ and $H = H' - D$. If $v \in V(H) \setminus A$, then $d^-_{H'}(v) \geq k$ and v has at most $k-1$ in-neighbors from D in H'. Then v has at least one in-neighbor in $V(H)$ and $d^-_H(v) \geq 1$. $\square$

The case of $k \geq \frac{\Delta}{2}$ the complexity of parameterization by b on DAGs is left open. However we can show that Dir-AKC is FPTon DAGs of maximum degree Δ, when parameterized by $\Delta + p$.

Theorem 6. *For any positive integers p and Δ,* Dir-AKC *can be solved in time $2^{O(\Delta p)} \cdot n^2 \log n$ for n-vertex DAGs of maximum degree at most Δ.*

Proof. Consider an instance of Dir-AKC with an n-vertex directed acyclic graph G. Without loss of generality we can assume that $b \leq p \leq n$.

We apply Lemma 2 for $q = p$. In time $2^{O(\Delta p)} \cdot n \log n$ we either obtain a solution or conclude that for any solution (A, H), H has size at least $p + 1$. If we obtain a solution, we return it. Suppose that we got a NO-answer. If $p = n$, then we return a NO-answer. Otherwise, we select a sink $t \in V(G)$ using the fact that any directed acyclic graph has at least one such vertex. Observe that we can assume that t is not an anchor in any solution. Also if t is included in a solution H of size at least $p + 1$, then $H - t$ is a solution of size at least p, because t is not joined by arcs with other vertices of H. Then we solve the instance $G - t$ of Dir-AKC recursively.

As each step is done in time $2^{O(\Delta p)} \cdot n \log n$ and the number of steps is at most n, the claim follows. $\square$

Let us remark that this result can be easily extended for any class of directed acyclic graphs $\mathcal{G}$ such that the corresponding class of underlaying graphs $\{G^* | G \in \mathcal{G}\}$ has (locally) bounded expansion by making use of the results by Dvorak et al. [12]. Finally, what happens when the input graph is planar? We know that the problem is NP-complete on planar graphs for fixed $k \geq 1$ and maximum degree $k + 2$. Is the problem FPT on planar directed graphs when parameterized by the size of the core p?

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